Lecture 2

Monday, 22 August 2022 11:29 AM

Recall:

- Notation:

  $N,\ \mathcal{S}_i,\ s_i \in \mathcal{S}_i,\ \mathcal{S},\ s \in \mathcal{S},\ \mathcal{S}_{-i},\ s_{-i},\ u_i/c_i$

- Equilibrium: a strategy profile $s^* \in \mathcal{S}$ is a pure strategy Nash equilibrium if each player chooses the best strategy wrt the other players, i.e., $\forall i \in N$, & for all $s_i' \in \mathcal{S}_i$,

  $$u_i(s_i', s_{-i}^*) \leq u_i(s_i^*, s_{-i}^*)$$

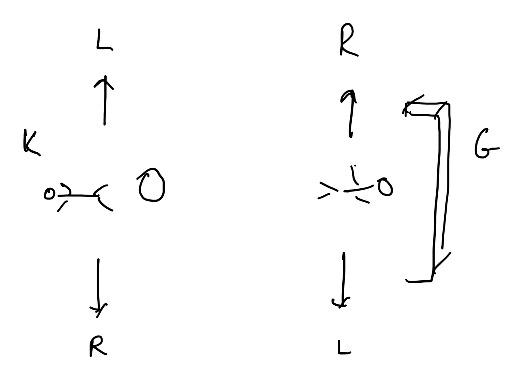
- **Lemma**: Let s^* be a NE in a game, and $\mathcal{S}' = \mathcal{S}_1' \times \dots \times \mathcal{S}_n'$ be the reduced game obtained from IRDS. Then $\forall i$, $s_i^* \in \mathcal{S}_i'$.

We saw examples of PSNE in the canteen game last time. Consider the following Penalty Shootout game

Game: Penalty Shootout.

G

K	L	R
L	10, −5	−5, 5
R	−5, 5	10, −5



Can check there is no PSNE.

We expand our definition to allow for randomized strategies. Instead of $s_i \in \mathcal{S}_i$ (called a pure strategy), each player can now play a distribution over $\mathcal{S}_i$ (called a <u>mixed strategy</u>)

Notation:

For player i, the set of mixed strategies

$$\Sigma_i = \left\{ x \in \mathbb{R}_+^{|\mathcal{S}_i|} : \sum_j x_j = 1 \right\}$$

& $\sigma_i \in \Sigma_i$ is a mixed strategy.

As previously we can define

$$\Sigma = \Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$$ as the set of all mixed strategy tuples.

& $\sigma \in \Sigma$ is a mixed strategy tuple / profile

Let σ_i be a mixed strategy for player i, & $a \in \mathcal{S}_i$ be a pure strategy. Then $\sigma_i(a)$ is the probability with which i plays a.

Example: In the PS game, say the kicker plays the mixed strategy $\sigma_K(L) = 1/3$, $\sigma_K(R) = 2/3$

And goalie plays $\sigma_G(L) = 1$, $\sigma_G(R) = 0$.

Utilities:

We've defined utilities for pure strategies $u_i : \mathcal{S} \to \mathbb{R}$.

We now extend this naturally to 'expected' utilities for mixed strategies $u_i : \Sigma \to \mathbb{R}$

where for $\sigma = (\sigma_1, \dots, \sigma_n)$,

$$u_i(\sigma) = \sum_{\substack{s \in \mathcal{S} \\ s = (s_1, \dots, s_n)}} u_i(s) \Pr(s) = \sum_{\substack{s \in \mathcal{S} \\ s = (s_1, \dots, s_n)}} u_i(s) \prod_{j=1}^{n} \sigma_j(s_j)$$

Thus, for the mixed strategy profile $\sigma_K = (1/3, 2/3)$, $\sigma_G = (1, 0)$,

$$u_K(\sigma) = 10 \times 1/3 + -5 \times 0 + -5 \times 2/3 + 10 \times 0 = 0$$

$$\& \quad u_G(\sigma) = -5 \times 1/3 + 5 \times 2/3 = 5/3$$

Q. What are the utilities of the players, if both play $(1/2, 1/2)$?

Notation for 2-player games:

We can separately represent the utilities of the 2 players in the penalty shootout game:

$$K = \begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{bmatrix} 10 & -5 \\ -5 & 10 \end{bmatrix} \end{array} \qquad G = \begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \end{array}$$

(note that G is transposed, when both represented together)

Further we can represent the mixed strategies of the 2 players as column vectors, x & y

Thus, $x = (1/3 \quad 2/3)^T$, $y = (1 \quad 0)^T$ is a strategy profile.

Then note that the utility of G for this strategy profile is simply $x^T G y$

& the utility for K for this strategy profile is $x^T K y$

(can check)

For 2-player (aka <u>bimatrix</u>) games, we will use this notation for mixed strategies. x is the mixed strategy for the row player, & y is the mixed strategy for the column player.

Defn (Mixed Strategy Nash Equilibria):

A mixed strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$ is a Nash Equilibrium if each player chooses the best (mixed) strategy wrt to the other players, i.e,

$\forall i \in N$, $\forall \sigma_i' \in \Sigma_i$,

$$u_i(\sigma_i', \sigma_{-i}^*) \leq u_i(\sigma_i^*, \sigma_{-i}^*)$$

Theorem (Nash 1950): Every finite game has an equilibrium in mixed strategies.

We can check that in the penalty shootout game, $\sigma_K^* = x^* = (1/2, 1/2)$, $\sigma_G^* = y^* = (1/2, 1/2)$ is a NE.

Suppose G plays $(1/2, 1/2) = y^*$. Then K's expected utility from its 2 strategies is $Ky^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence no matter what K plays, its utility cannot exceed $0 = x^{*T} K y^*$.

Suppose K plays $(1/2, 1/2) = x^*$. Then G's expected utility from its 2 strategies is $x^{*T} G = [5/2 \quad 5/2]$.

Support of NE: Let (A, B) be a bimatrix game, and (x^*, y^*) be a NE. Consider Ay^*, this is a column vector, giving the expected utility to the row player for each strategy. Now note that $(x_i^* > 0) \Rightarrow (Ay^*)_i$ must be maximum, else, x^* cannot be a best-response to y^*.

We say $\{i : x_i^* > 0\}$ is the support of x^*. Then x^* (and any NE) can only be supported on strategies that have maximum expected utility, given the strategy of the other players.

The Election Game: 2 political parties R & C, each must choose an issue to focus on in the coming elections.

C

R	(P)overty	(I)nfrastructure
(S)ecurity	3, −3	−1, 1
(E)conomy	−2, 2	1, −1

This game does not have a PSNE.

This is a **zero-sum game**: For each pure strategy, (and hence for each mixed strategy), the sum of utilities for the players is zero.

We will now see how to compute an equilibrium in a zero-sum game using linear programming.